# Anti-unification:
# Introduction, Applications, and Recent Results


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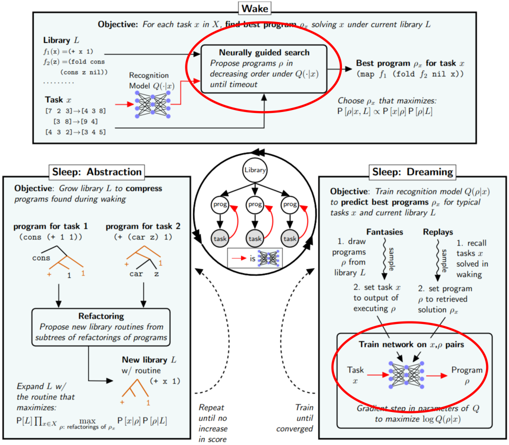
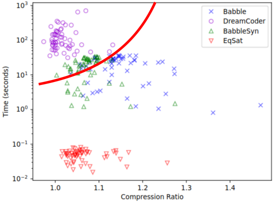
# DreamCoder: library learning modulo theory

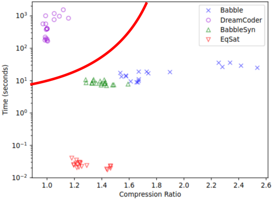



*DreamCoder: Bootstrapping Inductive Program Synthesis with Wake-Sleep Library Learning*, 2021, Ellis *et al.*, PLDI

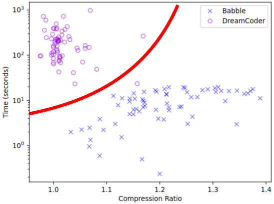
# Babble: library learning modulo theory



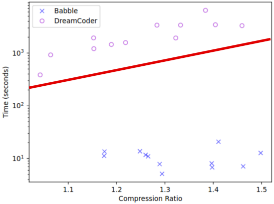

(a) List domain




(b) Physics domain




(c) Text domain

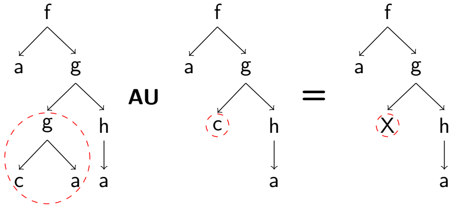



(d) Logo domain

*Babble: Learning Better Abstractions with E-Graphs and Anti-Unification*, Cao *et al.*, POPL

# What is it?

- Unification: is a process by which two symbolic expressions may be identified through variable replacement.
- Anti-unification: A process that derives from a set of symbolic expressions a new symbolic expression possessing certain commonalities shared between its members.




- Independently introduced by Plotkin and Reynolds in 1970.
  - "*A note on inductive generalization*" by G. D. Plotkin
  - "*Transformational systems and the algebraic structure of atomic formulas*" by J.C. Reynolds

# Anti-Unification: Basics

- Let $\Sigma$ be signature, $\mathcal{V}$ a countable set of variables, and $\mathcal{T}(\Sigma, \mathcal{V})$ a term algebra.
- (Unification) For $s, t \in \mathcal{T}(\Sigma, \mathcal{V})$:

  Does there exists a substitution $\sigma$ s.t. $s\sigma = t\sigma$?
- (Anti-Unification) For $s, t \in \mathcal{T}(\Sigma, \mathcal{V})$:

  Does there exists $g \in \mathcal{T}(\Sigma, \mathcal{V})$ and substitutions $\sigma_s$ and $\sigma_t$ s.t. $g\sigma_s = s$ and $g\sigma_t = t$?
- The term $g$ is referred to as a generalization of $s$ and $t$.
- While a substitution $\sigma$ such that $s\sigma = t\sigma$ may not exists, $x \in \mathcal{V}$ always generalizes $s$ and $t$ (typically...):

$$\sigma_s = \{x \mapsto s\} \ , \ \sigma_t = \{x \mapsto t\}$$

- Let's look at an example.

# Anti-Unification: Basics

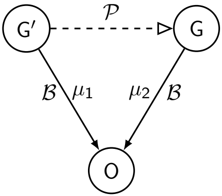
$$f(g(x, a)) \stackrel{?}{=} f(y)$$

- $\{x \leftarrow a\ ,\ y \leftarrow g(a, a)\}$ is a unifier.
  But, $\{y \leftarrow g(x, a)\}$ is more general.

$$f(g(b, a)) \triangleq f(g(a, a))$$

- $f(y)$ is a generalization, $\{y \leftarrow g(b, a)\}$ and $\{y \leftarrow g(a, a)\}$.
  But, $f(g(y, a))$ is more specific, $\{y \leftarrow b\}$ and $\{y \leftarrow a\}$
- Dual of most general unifier, least general generalization.
- Let $g_1$ and $g_2$ be generalizers of $t_1$ and $t_2$, then $g_1$ is less general then $g_2$, $g_2 \prec g_1$ if there exists $\mu$ s.t. $g_2\mu = g_1$.
- $g_1$ is least general if for every comparable term $g_2$, $g_2 \prec g_1$.

# A General Framework




| Generic | Concrete |
|---|---|
| $\mathcal{O}$ | $\mathcal{T}(\Sigma, \mathcal{V})$ |
| $\mathcal{M}$ | First-order substitutions |
| $\mathcal{B}$ | $\doteq$ (syntactic equality) |
| $\mathcal{P}$ | $\preceq$ : $s \preceq t$ if $s\sigma \doteq t$ for some $\sigma$ |

- **Goal:** from $O_1, O_2 \in \mathcal{O}$ (symbolic expressions) derive $G \in \mathcal{O}$ possessing certain commonalities shared by $O_1$ and $O_2$.
- **Specification:** define (a) a class of mappings $\mathcal{M}$ from $\mathcal{O} \to \mathcal{O}$, (b) a base relation $\mathcal{B}$ consistent with $\mathcal{M}$, and (c) a preference relation $\mathcal{P}$ consistent with $\mathcal{B}$.
- **Result:** G is a $\mathcal{B}$-generalization of $O_1$ and $O_2$ and most $\mathcal{P}$-preferred ("better" than $G'$).

# A General Framework

- A set $\mathcal{G} \subset \mathcal{O}$ is called $\mathcal{P}$-complete set of $\mathcal{B}$-generalizations of $O_1, O_2 \in \mathcal{O}$ if:
  - **Soundness:** Every $G \in \mathcal{G}$ is a $\mathcal{B}$-generalization of $O_1$ and $O_2$.
  - **Completeness:** For each $\mathcal{B}$-generalization $G'$ of $O_1$ and $O_2$, there exists $G \in \mathcal{G}$ such that $\mathcal{P}(G', G)$ (G is more preferred).
- Furthermore, $\mathcal{G}$ is minimal if:
  - **Minimality:** No distinct elements of $\mathcal{G}$ are $\mathcal{P}$-comparable: if $G_1, G_2 \in \mathcal{G}$ and $\mathcal{P}(G_1, G_2)$, then $G_1 = G_2$.
- Minimal Complete sets come in four Types:
  - **Unitary** ($\mathbf{1}$): $\mathcal{G}$ is a singleton,
  - **Finitary** ($\omega$): $\mathcal{G}$ is finite and contains at least two elements,
  - **Infinitary** ($\infty$): $\mathcal{G}$ is infinite,
  - **Nullary** ($\mathbf{0}$): $\mathcal{G}$ does not exist ( minimality and completeness contradict each other).
- Types are extendable to generalization problems.

# Complete sets of solutions

- Here are some examples for each category of complete sets:
  - **UNITARY:**
    - First-Order terms
    - High-Order patterns **(and friends)**
  - **FINITARY:**
    - FO terms, associative and/or commutative symbols
    - Unranked Terms and Hedges
    - FO terms, one symbol has a unit element
  - **INFINITARY:**
    - FO terms, idempotent symbols
    - FO terms, absorbing IJCAR 2024
  - **NULLARY:**
    - Semirings
    - FO terms, more than one symbol has a unit element
    - Simply typed lambda calculus , ACM TOCL 2024
    - Cartesian Combinators

# Rule-Based Algorithm

- $x : t \triangleq s$ is an anti-unification problem (AUP).
- A configuration is a triple $A; S; G$ where
  - $A$ is a set of AUPs **(Active)**
  - $S$ is a set of AUPs **(Solved)**
  - $G$ is a set of AUPs **(Generalization)**
- The initial state for an AUP $x : t \triangleq s$ is $\{x : t \triangleq s\};\ \emptyset;\ x$.
- Inference rules transform configurations into configurations.
- A configurations is final when no rules may be applied.

# Rule-Based AU: Examples

Dec: **Decomposition**

$$\{\underline{x : f(\overline{t_m}) \triangleq f(\overline{s_m})}\} \uplus A;\ S;\ G \Longrightarrow$$
$$\{y_m : t_m \triangleq s_m\} \cup A; S;\ G\{x \mapsto f(\overline{y_m})\},$$

where $y_1, \ldots, y_m$ are fresh variables

Sol: **Solve Rule**

$$\{x : t \triangleq s\} \uplus A;\ S;\ G \Longrightarrow A;\ \{x : t \triangleq s\} \cup S;\ G,$$

$head(t) \neq head(s)$ and $y$ is a fresh variable.

Mer: **Merge Rule**

$$A;\ \{x : t_1 \triangleq t_2, y : s_1 \triangleq s_2\} \uplus S;\ G \Longrightarrow$$
$$A;\ \{x : t_1 \triangleq t_2\} \cup S;\ G\,\{y \mapsto x\}\,,$$

$t_1 = s_1$ and $t_2 = s_2$.

# Rule-Based AU: Examples

$$\{x : f(g(a,c), h(b,a,b)) \triangleq f(a, h(a,a,a))\}; \emptyset; x$$

$$\Longrightarrow_{\mathsf{Dec}}$$

$$\{x_1 : g(a,c) \triangleq a, x_2 : h(b,a,b) \triangleq h(a,a,a)\}; \emptyset; f(x_1, x_2)$$

$$\Longrightarrow_{\mathsf{Sol}}$$

$$\{x_2 : h(b,a,b) \triangleq h(a,a,a)\}; \{x_1 : g(a,c) \triangleq a\}; f(x_1, x_2)$$

$$\Longrightarrow_{\mathsf{Dec}}$$

$$\{x_3 : b \triangleq a, x_4 : a \triangleq a, x_5 : b \triangleq a\}; \{x_1 : g(a,c) \triangleq a\}; f(x_1, h(x_3, x_4, x_5))$$

$$\Longrightarrow_{\mathsf{Dec}}$$

$$\{x_3 : b \triangleq a, x_5 : b \triangleq a\}; \{x_1 : g(a,c) \triangleq a\}; f(x_1, h(x_3, a, x_5))$$

$$\Longrightarrow_{\mathsf{Sol}^{\times 2}}$$

$$\emptyset; \{x_1 : g(a,c) \triangleq a, x_3 : b \triangleq a, x_5 : b \triangleq a\}; f(x_1, h(x_3, a, x_5))$$

$$\Longrightarrow_{\mathsf{mer}}$$

$$\emptyset; \{x_1 : g(a,c) \triangleq a, x_3 : b \triangleq a\}; f(x_1, h(x_3, a, x_3))$$

# Applications of Anti-unification

- Many applications are covered in the following Survey:

  *Anti-unification and Generalization: A Survey*, D.M. Cerna and T. Kutsia, IJCAI 2023 doi.org/10.24963/ijcai.2023/736

- Anti-unification is often used to build templates.

  *If objects match the template then they ought to behave similarly in a given situation.*

- Investigations have used anti-unification and similar techniques for inductive synthesis.

# Apps: Inductive Synthesis

- Second-order anti-unification for program Replay.

  *The Replay of Program Derivations*, R.W. Hasker, 1995, Thesis

- $\theta$-subsumption for building bottom clauses.

  *Inverse entailment and Progol*, S. Muggleton, 1995, NGCO

- Lggs used for recursive functional program synthesis.

  *IGOR II – an Analytical Inductive Functional Programming System*, M. Hofmann, 2010, PEPM

- Anti-unification for templating the recursion step.

  *Inductive Synthesis of Functional Programs: An Explanation Based Generalization Approach*, E. Kitzelmann U. Schmid, 2006, JMLR

- Flash-fill in Microsoft Excel.

  *Programming by Example using Least General Generalizations*, By M. Raza, S. Gulwani, N. Milic-Frayling, 2014, AAAI

# Applications:Bugs and Optimizations

- Extracting fixes from repository history.

  *Learning Quick Fixes from Code Repositories* by R. Sousa , *et al.*, 2021, SBES

- Templating bugs with corresponding fixes.

  *Getafix: Learning to Fix Bugs Automatically* By J. Bader, *et al.*, 2019, OOPSLA

- Templating configuration files to catagorize errors.

  *Rex: Preventing Bugs and Misconfiguration in Large Services Using Correlated Change Analysis* By Sonu Mehta, *et al.*, 2020, NSDI

- Optimization of recursion schemes for efficient parallelizability.

  *Finding parallel functional pearls: Automatic parallel recursion scheme detection in Haskell functions via anti-unification* By A. D. Barwell, C. Brown, K. Hammond, 2017, FGCS

# Applications:Theorem Proving

- Extraction of substitutions from substitution trees.

  *Higher-order term indexing using substitution trees* By B. Pientka, 2009, ACM TOCL

- Grammar compression and inductive theorem proving.

  *Algorithmic Compression of Finite Tree Languages by Rigid Acyclic Grammars*, By S. Eberhard, G. Ebner, S. Hetzl, 2017, ACM TOCL

- Generating SyGuS problems.

  *Reinforcement Learning and Data-Generation for Syntax-Guided Synthesis*, By J. Parsert and E. Polgreen, 2024, AAAI

# Anti-unification over Lambda Terms

- Let $\mathcal{B}$ be a set of base types and Types is the set of types inductively constructed from $\delta$ and $\rightarrow$.
- The set $\Lambda$ is constructed using the following grammar:

$$t ::= x \mid c \mid \lambda x.t \mid t_1 t_2$$

- A lambda term is a pattern if free variables only apply to distinct bound variables.
- $\lambda x.f(X(x), c)$ is a pattern, but $\lambda x.f(X(X(x)), c)$ and $\lambda x.f(X(x, x), c)$ are not.
- Anti-unification of an AUP $X(\vec{x}) : t \triangleq s$ often requires
  - $t$ and $s$ are of the same type ,
  - $t$ and $s$ are in $\eta$-long $\beta$-normal form,
  - and $X$ does not occur in $t$ and $s$.

# Anti-unification over Lambda Terms

- Calculus of Constructions, pattern fragment.

> *Unification and anti-unification in the calculus of construction* By F. Pfenning, 1991, LICS

- Anti-unification in $\lambda 2$ ($\mathcal{P}$ based on $\beta$-reduction).

> *Higher order generalization and its application in program verification*, Lu *et al.*, 2000, AMAI

- Pattern Anti-unification in simply-typed $\lambda$-calculus.

> *Higher-order pattern anti-unification in linear time*, A. Baumgartner *et al.*, 2017, JAR

- Top-maximal shallow, simply-typed $\lambda$-calculus.

> *A generic framework for higher-order generalization*, D. Cerna and T. Kutsia, 2019, FSCD

- $\lambda$-calculus with recursive let expressions.

> *Towards Fast Nominal Anti-unification of Letrec-Expressions*, M. Schmidt-Schauß, D. Nantes-Sobrinho *et al.*, 2023, CADE

# Rules: Pattern Anti-unification

Dec: **Decomposition**

$\{\underline{X(\vec{x}) : \mathrm{h}(\overline{t_m}) \triangleq} \mathrm{h}(\overline{s_m})\} \uplus A;\ S;\ \sigma \Longrightarrow$
$\{Y_m(\vec{x}) : t_m \triangleq s_m\} \cup A; S;\ G\{X \mapsto \lambda\vec{x}.h(\overline{Y_m(\vec{x})})\}$,
where $h$ is constant or $h \in \vec{x}$, and $\overline{Y_m}$ are fresh variables of the appropriate types.

Abs: **Abstraction Rule**

$\{X(\vec{x}) : \lambda y.t \triangleq \lambda z.s\} \uplus A;\ S;\ \sigma \Longrightarrow \{X'(\vec{x}, y) : t \triangleq s\{z \mapsto y\}\} \cup A;\ S;\ G\,\{X \mapsto \lambda\vec{x}, y.X'(\vec{x}, y)\}$,
where $X'$ is a fresh variable of the appropriate type.

# Extensions: Lambda Terms

Sol: **Solve Rule**

$\{X(\vec{x}) : t \triangleq s\} \uplus A;\ S;\ \sigma \Longrightarrow$
$A;\ \{Y(\vec{y}) : t \triangleq s\} \cup S;\ G\{X \mapsto \lambda\vec{x}.Y(\vec{y})\}$,

where t and s are of a base type, $head(t) \neq head(s)$ or $head(t) = head(s) = \mathrm{h} \notin \vec{x}$. The sequence $\vec{y}$ is a subsequence of $\vec{x}$ consisting of the variables that appear freely in $t$ or in $s$, and $Y$ is a fresh variable of the appropriate type.

Mer: **Merge Rule**

$A;\ \{X(\vec{x}) : t_1 \triangleq t_2, Y(\vec{y}) : s_1 \triangleq s_2\} \uplus S;\ \sigma \Longrightarrow A;\ \{X(\vec{x}) : t_1 \triangleq t_2\} \cup S;\ G\{Y \mapsto \lambda\vec{y}.X(\vec{x}\pi)\}$,

where $\pi : \{\vec{x}\} \to \{\vec{y}\}$ is a bijection, extended as a substitution with $t_1\pi = s_1$ and $t_2\pi = s_2$.

# Pattern Anti-unification: Example

$$\{X : \lambda x, y.f(u(g(x), y), u(g(y), x)) \triangleq \lambda x', y'.f(h(y', g(x')), h(x', g(y')))\};$$
$$\emptyset; X \Longrightarrow_{\mathsf{Abs}\times 2}$$
$$\{X'(x, y) : f(u(g(x), y), u(g(y), x)) \triangleq f(h(y, g(x)), h(x, g(y)))\};\ \emptyset;$$
$$\lambda x, y.X'(x, y) \Longrightarrow_{\mathsf{Dec}}$$
$$\{Y_1(x, y) : u(g(x), y) \triangleq h(y, g(x)), Y_2(x, y) : u(g(y), x) \triangleq h(x, g(y))\};\ \emptyset;$$
$$\lambda x, y.f(Y_1(x, y), Y_2(x, y)) \Longrightarrow_{\mathsf{Sol}}$$
$$\{Y_2(x, y) : u(g(y), x) \triangleq h(x, g(y))\}; \{Y_1(x, y) : u(g(x), y) \triangleq h(y, g(x))\};$$
$$\lambda x, y.f(Y_1(x, y), Y_2(x, y)) \Longrightarrow_{\mathsf{Sol}}$$
$$\emptyset;\ \{Y_1(x, y) : u(g(x), y) \triangleq h(y, g(x)), Y_2(x, y) : u(g(y), x) \triangleq h(x, g(y))\};$$
$$\lambda x, y.f(Y_1(x, y), Y_2(x, y)) \Longrightarrow_{\mathsf{Mer}}$$
$$\emptyset;\ \{Y_1(x, y) : u(g(x), y) \triangleq h(y, g(x))\};\ \lambda x, y.f(Y_1(x, y), Y_1(y, x))$$

# Friends of Patterns

- While useful, patterns are quite inexpressive.

  *Functions-as-Constructors Higher-Order Unification*, T. Libal and D. Miller, 2016, FSCD

  - Restricted terms occur as arguments to free variables.
  - Restricted terms are inductively constructed from bound variables and constant symbols with arity $> 0$.
  - Arguments cannot be subterms of each other.
    - $X(f(x), y)$ is ok, but not $X(f(x), x)$.
  - Arguments cannot be proper subterms of each other.
    - $g(X(f(x), y), Y(f(x), z))$ is ok, but not $g(X(f(x), y), Y(x))$.
- Unitary, but is Finitary without variable restrictions.
- Anti-unification is Unitary without most restrictions.

# Friends of Patterns

- Rules construct Top-maximal Shallow Generalizations.
  - $\lambda x.f(X(x))$ is preferred to $\lambda x.X(f(x))$ when possible.
  - $\lambda x.f(X(X(x)))$ or $\lambda x.f(X(Y(x)))$ not allowed.
- Only the Solve rule changes:

Sol: **Solve**

$$\{X(\vec{x}) : t \triangleq s\} \uplus A;\ S;\ r \Longrightarrow A;\ \{Y(y_1, \ldots, y_n) : (C_t\, y_1 \cdots y_n) \triangleq (C_s\, y_1 \cdots y_n)\} \cup S;\ r\{X \mapsto \lambda\vec{x}.Y(q_1, \ldots, q_n)\},$$

where $t$ and $s$ are of a basic type, $head(t) \neq head(s)$, $q_1, \ldots, q_n$ are distinct subterms of $t$ or $s$, $C_t$ and $C_s$ are terms such that $(C_t\, q_1 \cdots q_n) = t$ and $(C_s\, q_1 \cdots q_n) = s$, $C_t$ and $C_s$ do not contain any $x \in \vec{x}$, and $Y$, $y_1, \ldots, y_n$ are distinct fresh variables of the appropriate type.

- Pattern if the $q_1, \ldots, q_n \in \vec{x}$, and $C_t = \lambda\vec{x}.t$ and $C_s = \lambda\vec{x}.s$.

# Anti-Unification beyond Patterns

- Not every choice of $C_s$ and $C_t$ will result in a Unitary variant.
- Inconsistent choices for $C_s$ and $C_t$ can result in the computation of non-lggs.
- In particular how the $q_i s$ are chosen matters:
  - $q_i s$ must match a selection condition.
  - $q_i s$ must occur in both terms.
  - $q_i s$ must not be positionally ordered within the terms.
- These conditions allowed us to define 4 Unitary variants.

# Anti-Unification beyond Patterns

- Projection Anti-Unification:
  - $q_1 = t,\ q_2 = s,\ C_t = \lambda z_1, z_2.z_1,\ C_s = \lambda z_1, z_2.z_2.$
- Common Subterms Anti-Unification:
  - $q_i s$ position maximal common subterms.
  - $C_t = \lambda y_1, \ldots, y_n.\, t[p_1 \mapsto y_1] \cdots [p_m \mapsto y_n]$
  - $C_s = \lambda y_1, \ldots, y_n.\, s[l_1 \mapsto y_1] \cdots [l_m \mapsto y_n]$
- Restricted Function-as-constructor Anti-Unification:
  - $q_i s$ position maximal common subterms minus those which break the Local variable condition.
  - $C_t$ and $C_s$ are the same.
- Function-as-constructor Anti-Unification:
  - $q_i s$ position maximal common subterms minus those which break the Local/Global variable conditions.
  - $C_t$ and $C_s$ are the same.
- Other variants are definable (based on the selection condition).

# Anti-Unification beyond Patterns: Example

$$\{X : \lambda x. f(h_1(g(g(x)), a, b), h_2(g(g(x)))) \triangleq$$
$$\lambda y. f(h_3(g(g(y)), g(y), a), h_4(g(g(y))))\};\ \emptyset;\ X$$

$\Longrightarrow_{\text{Abs}}$

$$\{X'(x) : f(h_1(g(g(x)), a, b), h_2(g(g(x)))) \triangleq$$
$$f(h_3(g(g(x)), g(x), a), h_4(g(g(x))))\}; \emptyset;\ \lambda x. X'(x)$$

$\Longrightarrow_{\text{Dec}}$

$$\{Z_1(x) : h_1(g(g(x)), a, b) \triangleq h_3(g(g(x)), g(x), a),$$
$$Z_2(x) : h_2(g(g(x))) \triangleq h_4(g(g(x))\};\ \emptyset;$$
$$\lambda x. f(Z_1(x), Z_2(x))$$

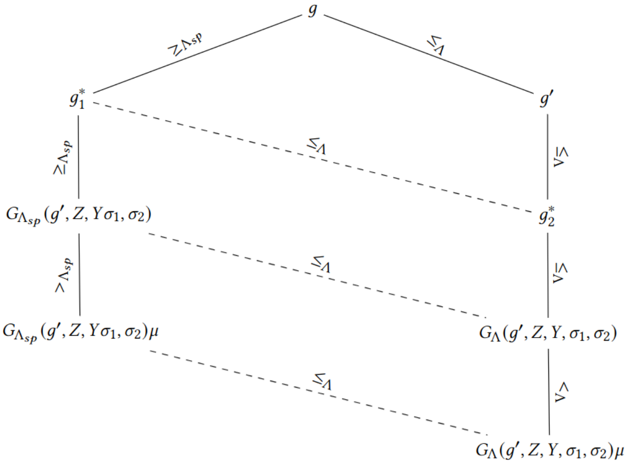
$\Longrightarrow_{\text{Sol-RFC}}$

# Anti-Unification beyond Patterns: Example

$$\{Z_2(x) : h_2(g(g(x))) \triangleq h_4(g(g(x)))\};$$
$$\{Y_1(y_1) : h_1(g(y_1), a, b) \triangleq h_3(g(y_1), y_1, a)\};$$
$$\lambda x.f(Y_1(g(x)), Z_2(x))$$
$$\Longrightarrow_{\text{Sol-RFC}}$$
$$\emptyset;\ \{Y_1(y_1) : h_1(g(y_1), a, b) \triangleq h_3(g(y_1), y_1, a),$$
$$Y_2(y_2) : h_2(y_2) \triangleq h_4(y_2)\};$$
$$\lambda x.f(Y_1(g(x)), Y_2(g(g(x)))).$$

- Extending this idea to higher-type theories such as the calculus of constructions (COC) has yet to be considered?
- Beneficial for proof generalization.
- What happens when the terms are no longer shallow?

# Deep Lambda Terms: Nullarity




- $\lambda x.\lambda y.f(x) \triangleq \lambda x.\lambda y.f(y)$ has no solution set.
- $\lambda x.\lambda y.f(Z(x,y)) < \lambda x.\lambda y.f(Z(Z(x,y), Z(x,y))) < \cdots$

# Deep Lambda Terms: Nullarity

- Its pattern generalization is $g^p = \lambda x.\lambda y.f(Z(x,y))$.
- A generalization more specific $g^p$ is pattern-derived

### Definition

Let $g$ be pattern-derived. Then $g$ is *tight* if for all $W \in \mathcal{FV}(g)$:

1) $g\{W \mapsto \lambda\overline{b_k}.b_i\} \notin \mathcal{G}(s,t)$, if $W$ has type $\overline{\gamma_k} \to \gamma_i$ and for $1 \leq i \leq k$ and $\gamma_i \in \mathcal{B}$, and
2) For $(\sigma_1, \sigma_2) \in \mathcal{GS}(s,t,g)$, $g\{W \mapsto t_1\}, g\{W \mapsto t_2\} \notin \mathcal{G}(s,t)$ where $t_1 = W\sigma_1$, $t_2 = W\sigma_2$.

# Deep Lambda Terms: Nullarity

**Definition**

Let $g = \lambda x.\lambda y.f(Z(\overline{s_m}))$ be a tight generalization of $s \triangleq t$ where

1) $Z$ has type $\overline{\delta_m} \to \alpha$ for $1 \leq i \leq m$, and $s_i$ has type $\delta_i$.
2) $(\sigma_1, \sigma_2) \in \mathcal{GS}(s, t, g)$ such that $Z\sigma_1 = r_1$ and $Z\sigma_2 = r_2$,
3) $r_1$ and $r_2$ are of type $\overline{\delta_m} \to \alpha$, and
4) $Y$ such that $Y \notin \mathcal{FV}(g)$ and has type $\alpha \to \alpha \to \alpha$.

Then the *g-pseudo-pattern*, denoted $G(g, Z, Y, \sigma_1, \sigma_2)$, is

$$g\{Z \mapsto \lambda\overline{b_m}.Y(r_1(\overline{b_m}), r_2(\overline{b_m}))\} = \lambda x.\lambda y.f(Y(r_1(\overline{q_m}), r_2(\overline{q_m})))$$

where for all $1 \leq i \leq m$, $q_i = s_i\{Z \mapsto \lambda\overline{b_m}.Y(r_1(\overline{b_m}), r_2(\overline{b_m}))\}$.

- Essentially, we regularized the structure of the generalizations.

# Deep Lambda Terms: Nullarity

**Theorem**

*For anti-unification of simply-typed lambda terms is nullary.*

**Proof.**

Let us assume that $C \subseteq \mathcal{G}(s,t)$ is minimal and complete. We know $C$ contains a pattern-derived generalization $g$. Observe that $g$ can be transformed into an tight generalization $g'$ that is also pattern-derived. We can derive a pseudo-pattern generalization $g''$ of $g'$. Finally, $g^* = g''\{Y \mapsto \lambda w_1.\lambda w_2.Y(Y(w_1,w_2),Y(w_1,w_2))\}$ is strictly more specific than $g''$. This implies that $g <_{\mathcal{L}} g^*$, entailing that $C$ is not minimal. □

- Result extendable to non-shallow fragments.

| |
|---|
| *One or nothing: Anti-unification over the simply-typed lambda calculus*, D. Cerna and M. Buran, 2024, ACM TOCL (just accepted). |

# Equational Anti-unification

- Anti-unification over commutative theories.

  *Unification, weak unification, upper bound, lower bound, and generalization problem*, F. Baader, 1991, RTA

- Grammar for a **complete** set of E-generalizations:

  *E-generalization using grammars*, J. Burghardt, 2005, AI

- Minimal complete set of AC-generalizations.

  *A modular order-sorted equational generalization algorithm*, M. Alpuente *et al.*, 2014, Inf. Comput.

- Minimal complete set of I-generalizations.

  *Idempotent anti-unification*, D. Cerna and T. Kutsia, 2020, ACM TOCL

- Nullarity of $U^2$-generalization.

  *Unital anti-unification: Type and algorithms*, M. D. Cerna and T. Kutsia, 2020, FSCD

# E-generalization: Important, but Explosive

- Many equational theories are not well behaved:

| Problem | Theory | Type |
|---|---|---|
| $f(a,b) \triangleq f(b,a)$ | $f(x,x) = x,$ | $\infty$ |
| $g(\varepsilon_f, f(a, h(\varepsilon_f))) \triangleq g(f(h(\varepsilon_f), a), \varepsilon_f)$ | $f(\varepsilon_f, x) = f(x, \varepsilon_f) = \varepsilon_f$ | $\infty$ |
| $0 \triangleq 1$ | Semirings | 0 |
| $a \triangleq b$ | f(a)=a, f(b)=b | 0 |

- Even when there are *least general generalizations*,
- are the majority of them useful? $f(f(f(\cdots f(x)\cdots)))$
- Though, not all theories behave badly....

# Equantional Anti-unification: A and C

- AC-Anti-unification is finitary.
  - Though the minimal complete set may have an exponential number generalizations.
- Assuming that $f$ is associative:

$$X : f(a, a, b, b) \triangleq f(a, b, b) \quad \text{(Flattened for Readability)}$$

- Note that there are many ways to decompose the problem:

$$X_1 : a \triangleq a \qquad X_2 : f(a, b, b) \triangleq f(b, b) \qquad (1)$$

$$X_1 : a \triangleq f(a, b) \qquad X_2 : f(a, b, b) \triangleq b \qquad (2)$$

$$X_1 : f(a, a, b) \triangleq a \qquad X_2 : b \triangleq f(b, b) \qquad (3)$$

$$X_1 : f(a, a) \triangleq a \qquad X_2 : f(b, b) \triangleq f(b, b) \qquad (4)$$

# Equantional Anti-unification: A and C

- If we continue this decomposition the lggs are:

$$g_1 = f(X_1, b, b) \quad g_2 = f(a, X_2, b)$$

- $g_1$ and $g_2$ are $\prec_A$-incomparable, and form the minimal complete set for the terms $f(a, a, b, b)$ and $f(a, b, b)$.
- To compute the minimal complete set modulo associativity we extend the syntactic algorithm by the following rules:

# Equantional Anti-unification: A Rules

Dec-A-L: **Associative Decomposition Left**

$$\{X : f(t_1, \dots, t_k, t_{k+1} \dots, t_n) \triangleq f(s_1, s_2 \dots, s_m)\} \uplus A;\ S;\ \sigma \Longrightarrow$$
$$\{Y_1 : f(t_1, \dots, t_k) \triangleq s_1,\ \ Y_2 : f(t_{k+1} \dots, t_n) \triangleq$$
$$f(s_2 \dots, s_m)\} \cup A;\ S;\ G\{X \mapsto f(Y_1, Y_2)\},$$

where f is associative, $n, m \geq 2$, $1 \leq k \leq n-1$, and $Y_1$ and $Y_2$ are fresh variables.

Dec-A-R: **Associative Decomposition Right**

$$\{X : f(t_1, t_2 \dots, t_n) \triangleq f(s_1, \dots, s_k, s_{k+1} \dots, s_m)\} \uplus A;\ S;\ \sigma \Longrightarrow$$
$$\{Y_1 : t_1 \triangleq f(s_1, \dots, s_k),\ \ Y_2 : f(t_2 \dots, t_n) \triangleq$$
$$f(s_{k+1} \dots, s_m)\} \cup A;\ S;\ G\{X \mapsto f(Y_1, Y_2)\},$$

where f is associative, $n, m \geq 2$, $1 \leq k \leq m-1$, and $Y_1$ and $Y_2$ are fresh variables

# Equantional Anti-unification: A and C

- Similarly one can define Commutative anti-unification.
- We assume that $f$ is commutative:

$$X : f(a, f(a, b)) \triangleq f(b, f(b, a))$$

- There are only two ways to decompose:

$$X_1 : a \triangleq b \qquad X_2 : f(a, b) \triangleq f(b, a) \tag{5}$$

$$X_1 : a \triangleq f(b, a) \qquad X_2 : f(a, b) \triangleq b \tag{6}$$

- Furthermore, one of the possible decompositions is syntactic.

# Equantional Anti-unification: A and C

- Continuing this decomposition we get two lggs:

$$g_1 = f(x, f(a, b)) \quad g_2 = f(x, f(x, y))$$

- Observe, $g_1$ and $g_2$ are $\prec_C$-incomparable and form the minimal complete set.
- To computing the minimal complete set modulo commutatively we extend the syntactic algorithm by the following rule:

# Equantional Anti-unification: A and C

Dec-C: **Commutative Decomposition**

$\{X : f(t_1, t_2) \triangleq f(s_1, s_2)\} \uplus A;\ S;\ \sigma \Longrightarrow \{Y_1 : t_1 \triangleq s_i,\ \ Y_2(\vec{x}) : t_2 \triangleq s_{(i \mod 2)+1}\} \cup A;\ S;\ G\{X \mapsto f(Y_1, Y_2)\}$,

where $f$ is commutative, $i \in \{1, 2\}$, and $Y_1$ and $Y_2$ are fresh variables

- We can combine the A and C inference rules and construct an even more flexible anti-unification algorithm.
- This combined anti-unification problem is still Finitary.
- $f(a, a, b) \triangleq f(a, b, b)$ has solutions $f(a, b, x)$ and $f(x, x, y)$.

# Selection Heuristics

- How to deal with the explosion?
  - Alignment and Rigidity functions
  - Skeletons
  - beam search
  - Syntactic restriction
- **Recent Direction:**
  - Should the preference and base relations be Crisp?
  - Are most lggs too distant from the generalized terms to be generalizations?
- Is **similarity** and **quantitative** anti-unification a fix?

*A Framework for Approximate Generalization in Quantitative Theories*, T. Kutsia and C. Pau, 2022, FSCD

# Future Work

- Investigating the above questions
- New applications for anti-unification
- Developing methods for combining anti-unification algorithms for disjoint equational theories
- Characterization of classes of equational theories that exhibit similar behavior and properties
- Studying computational complexity and optimizations.